Quantum state redistribution based on a generalized decoupling

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We develop a simple protocol for a one-shot version of quantum state redistribution, which is the most general two-terminal source coding problem. The protocol is simplified from a combination of protocols for the fully quantum reverse Shannon and fully quantum Slepian-Wolf problems, with its time-reversal symmetry being apparent. When the protocol is applied to the case where the redistributed states have a tensor power structure, more natural resource rates are obtained.

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Introduction–Quantum information theory may be understood in terms of interconversion between various resources [1, 2]. In this resource framework, the well-known quantum teleportation [3] can be regarded as a process in the simulation of noiseless quantum channels using entanglement plus noiseless classical channels. Significant efforts and progress have been made in the unification of quantum information theory[4, 5, 6, 7]; many important results in quantum information theory, such as one-way entanglement distillation [1], state merging [8], fully quantum Slepian-Wolf theorem (FQSW) [6], fully quantum reverse Shannon theorem (FQRS) [6, 9], and quantum channel capacities [10, 11, 12, 13, 14], can be derived in the framework of quantum state redistribution (QSR) [4, 5].

The one-shot version of QSR refers to a communication scenario where Alice and Bob share a quantum state φ_{CAB} in which Alice holds AC and Bob holds B. The shared state φ_{CAB} can be viewed as the reduced state of $|\varphi_{CABR}\rangle$ where R is an inaccessible reference system. The task is to redistribute C to Bob while trying to keep the whole pure state unchanged [see FIG. 1]. As in the classical information theory, it is interesting to consider the case where Alice, Bob and the reference system share many copies of $|\varphi_{CABR}\rangle$ and the task is to redistribute all C from Alice to Bob. Since each copy of the state is identical and independently distributed this is often called the i.i.d. case. To accomplish the task, protocols are allowed to use noiseless quantum communication and entanglement. Minor imperfections in the final state are tolerable provided that they vanish in the asymptotic region, i.e., when the number of copies goes to infinity. The minimal resources needed per copy in the asymptotic region are important and useful information-theoretic quantities.

QSR was first studied by Luo and Devetak[15], where the necessary resources needed per copy in the asymptotic region of the i.i.d. case were given. Later, Yard and Devetak indicated that the necessary resources are also sufficient [5], i.e., the redistribution of each copy can be accomplished by sending $Q = I\left(C;R\,|B\,\right)/2$ qubits to Bob and consuming $E = \left[I(A;C) - I(B;C)\right]/2$ ebits in

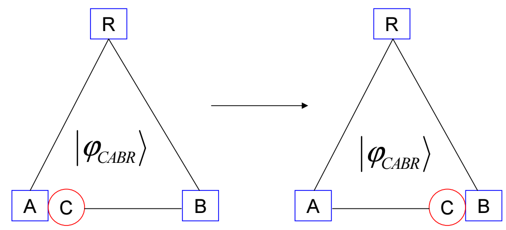

FIG. 1: (Color online). Quantum state redistribution. Initially A and C is held by Alice, B is held by Bob and R is the reference system. The task is to redistribute C from Alice to Bob. If Bob's side information B is viewed as a part of the reference system, it is reduced to FQRS problem; If Alice's side information A is viewed as a part of the reference system, it is reduced to FQSW problem.

the asymptotic region, here $I\left(C;R\,|B\,\right)$ is quantum conditional mutual information and $I(A;C)$ is quantum mutual information defined on $|\varphi_{CABR}\rangle$. If E is negative, instead of be consumed, it is distilled. However in the process of obtaining the resource rates, coherent channels [16] plus the cancelation lemma were used, which leads the process to be more complicated [5].

In this paper, we first develop a simple protocol for a one-shot version of QSR, which is simplified from a direct combination of protocols for FQRS and FQSW problems [see FIG. 2]. The simplification can be done based on a generalized decoupling. The application of our protocol in the i.i.d. case shows the redistribution of each copy can be accomplished by sending $Q = I\left(C;R\,|B\,\right)/2$ qubits from Alice to Bob, consuming $E_1 = I(A;C)/2$ ebits and distilling $E_2 = I(B;C)/2$ ebits at the same time in the asymptotic region. This result looks more natural and the net entanglement consumed is also $E = E_1 - E_2$ ebits per copy [4, 5].

QSR: one-shot version– We first give some notations. The density operator of state $|\varphi\rangle$ is denoted by φ. We use d_A to denote the dimension of A and $U \cdot \rho$ to denote the adjoint action of U on ρ, i.e., $U \cdot \rho = U\rho U^{-1}$. The

trace norm of ρ is defined as $\|\rho\|_1 = Tr\sqrt{\rho^\dagger \rho}$.

Lemma 1 *(lemma 2.2 of [2]) For two pure states $|\mu_{AB}\rangle$ and $|\nu_{AC}\rangle$, if the corresponding reduced states μ_A and ν_A satisfy $\|\mu_A - \nu_A\|_1 \leq \epsilon$, there exists an isometry $K_{B\to C}$ such that $\|K_{B\to C}\cdot \mu_{AB} - \nu_{AC}\|_1 \leq 2\sqrt{\epsilon}$.*

Theorem 2 *(generalized decoupling theorem) For any given density operators ω_{CF} and ψ_{CE} in which $C = C_1C_2C_3$, there exists a unitary operation U on C such that*

$$\|Tr_{C_2C_3}[U\cdot\omega_{CF}] - \pi_{C_1}\otimes\omega_F\|_1^2 \leq 2\alpha, \tag{1}$$

$$\|Tr_{C_1C_3}[U\cdot\psi_{CE}] - \pi_{C_2}\otimes\psi_E\|_1^2 \leq 2\beta, \tag{2}$$

where π_{C_1} and π_{C_2} are maximal mixed states, ω_F and ψ_E are the reduced states from ω_{CF} and ψ_{CE}. The upper bounds are defined as $\alpha = d_C d_F Tr\left(\omega_{CF}\right)^2/d_{C_2C_3}^2$ and $\beta = d_C d_E Tr\left(\psi_{CE}\right)^2/d_{C_1C_3}^2$.

Proof. The decoupling theorem [5, 6, 17, 18] states

$$\int\limits_{\mathbb{U}(C)} \|Tr_{C_2C_3}[U\cdot\omega_{CF}] - \pi_{C_1}\otimes\omega_F\|_1^2\, dU \leq \alpha, \tag{3}$$

$$\int\limits_{\mathbb{U}(C)} \|Tr_{C_1C_3}[U\cdot\psi_{CE}] - \pi_{C_2}\otimes\psi_E\|_1^2\, dU \leq \beta. \tag{4}$$

These inequalities indicate more than one half of the unitary operators U on C satisfying (1) and more than one half of the unitary operators U on C satisfying (2), so there exists a unitary operation U on C such that both (1) and (2) are satisfied at the same time. ∎

We introduce two reference states $|\hat{\varphi}_{CABR}\rangle$ and $|\check{\varphi}_{CABR}\rangle$ in discussing the redistribution of $|\varphi_{CABR}\rangle$ and define

$$\gamma_1 = 2\left\|\varphi_{CABR} - \hat{\varphi}_{CABR}\right\|_1, \tag{5a}$$

$$\gamma_2 = 2\left\|\varphi_{CABR} - \check{\varphi}_{CABR}\right\|_1. \tag{5b}$$

Assume that $C = C_1C_2C_3$, A'' and B' are the duplicates of A and B, C' and C'' are the duplicates of C. The generalized decoupling theorem 2 ensures that we are able to approximately decouple C_2 from BR in $|\hat{\varphi}_{CABR}\rangle$ and approximately decouple C_1 from AR in $|\check{\varphi}_{CABR}\rangle$ by the same operation on C. Combining with the lemma 1 and following the similar deduction in [6], it can be seen that there exist the unitary operation U on C, the isometries $W_{C_1C_3A\to A_2C''A''}$, and $V_{C_2C_3B\to B_1C'B'}$ such that

$$\|(W\circ U)\cdot\hat{\varphi}_{CABR} - \Phi_{C_2A_2}\otimes\hat{\varphi}_{C''A''BR}\|_1 \leq \eta_1, \tag{6}$$

$$\|(V\circ U)\cdot\check{\varphi}_{CABR} - \Phi_{C_1B_1}\otimes\check{\varphi}_{C'AB'R}\|_1 \leq \eta_2, \tag{7}$$

where $\Phi_{C_2A_2}$ and $\Phi_{C_1B_1}$ are the maximally entangled pure states, $\hat{\varphi}_{C''A''BR}$ and $\check{\varphi}_{C'AB'R}$ are the same pure states as $\hat{\varphi}_{CABR}$ and $\check{\varphi}_{CABR}$ respectively. The upper bounds η_1 and η_2 are defined as

$$\eta_1 = 2\sqrt[4]{2d_C d_{BR} Tr\left(\hat{\varphi}_{CBR}\right)^2/d_{C_1C_3}^2}, \tag{8a}$$

$$\eta_2 = 2\sqrt[4]{2d_C d_{AR} Tr\left(\check{\varphi}_{CAR}\right)^2/d_{C_2C_3}^2}. \tag{8b}$$

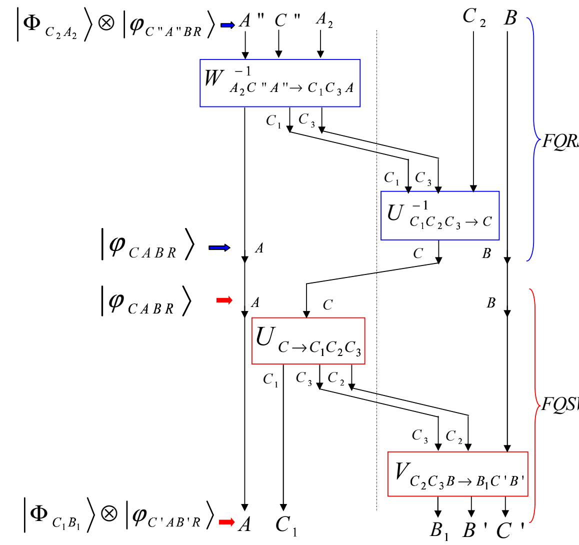

FIG. 2: (Color online) A non-optimal protocol for one-shot version of QSR that is a direct combination of protocols for FQRS and FQSW problems. The decoding operation U^{-1} in FQRS is the inversion of the encoding operation U in FQSW, so their effect cancels out.

Making use of the triangle inequality, we have from (6) and (7)

$$\|(W\circ U)\cdot\varphi_{CABR} - \Phi_{C_2A_2}\otimes\varphi_{C''A''BR}\|_1 \leq \Delta_1, \tag{9}$$

$$\|(V\circ U)\cdot\varphi_{CABR} - \Phi_{C_1B_1}\otimes\varphi_{C'AB'R}\|_1 \leq \Delta_2, \tag{10}$$

where $\Delta_i = \gamma_i + \eta_i$, $\varphi_{C''A''BR}$ and $\varphi_{C'AB'R}$ are the same pure state as φ_{CABR}.

In Fig. 2 we depict a protocol for redistributing $|\varphi_{C''A''BR}\rangle$ where Alice holds $C''A''$ and Bob holds B, i.e., redistributing C'' from Alice to Bob. This protocol is non-optimal since it cannot achieve the optimal resource rates in the i.i.d. case. However, its simplified version shown in Fig. 3 is optimal. The non-optimal protocol in Fig. 2 consists of three steps. The first step is to redistribute C'' from Alice to Bob via the protocol for FQRS problem [6] where Bob's side information B is treated as a part of the reference system. In this step, the maximally entangled state $\Phi_{C_2A_2}$ is consumed and the system C_1C_3 is transmitted from Alice to Bob via noiseless quantum channels. According to (9) the system is in $|\varphi_{CABR}\rangle$ after completing this step if $\Delta_1 = 0$. The second step is to send $C = C_1C_2C_3$ from Bob back to Alice via noiseless quantum channels. The third step is the process that redistributes C from Alice to Bob via the protocol for FQSW problem [6] where Alice's side information A is treated as a part of the reference system. In this step the system C_2C_3 is transmitted from Alice to Bob via noiseless quantum channels. According to (10) the system is in $|\Phi_{C_1B_1}\rangle\otimes|\varphi_{C'AB'R}\rangle$ after this step if $\Delta_1 = \Delta_2 = 0$, i.e., the maximally entangled state $\Phi_{C_1B_1}$ is distilled and the redistribution is perfectly accomplished since $|\varphi_{C'AB'R}\rangle$ is the same state as $|\varphi_{C''A''BR}\rangle$ but with A on Alice's

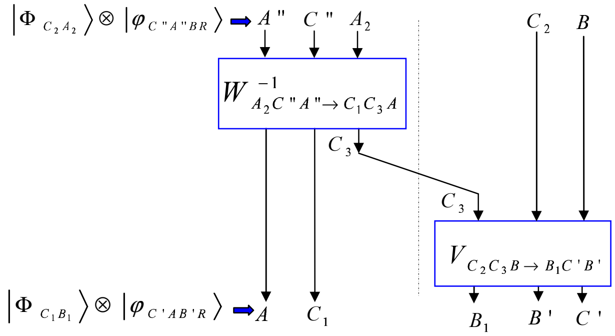

FIG. 3: (Color online) Our protocol for one-shot version of QSR. The encoding operation by Alice is W^{-1} and the decoding operation by Bob is V. The reverse redistribution uses V^{-1} and W as the encoding and decoding operations. The time-reversal symmetry of the protocol is apparent. The states on the left are obtained if $\Delta_1+\Delta_2$ is zero.

side and $C'B'$ on Bob's side.

For the above described non-optimal protocol, the decoding operation $U^{-1}_{C_1C_2C_3\to C}$ in the first step is the inversion of the encoding operation $U_{C\to C_1C_2C_3}$ in the third step, which is the result of the generalized decoupling theorem. So their effects cancel out and the whole process can be simplified. Our protocol for redistributing C'' in $|\varphi_{C''A''BR}\rangle$ from Alice to Bob is just the simplified version of the above one and it consists of three steps [see FIG. 3]:

1. Alice first implements the encoding operation $W^{-1}_{A_2C''A''\to C_1C_3A}$ on the initial pure state $\Phi_{C_2A_2}\otimes \varphi_{C''A''BR}$ where Alice holds $A_2C''A''$ and Bob holds C_2B.

2. Alice transmits C_3 to Bob via noiseless quantum channels.

3. Bob makes the decoding operation $V_{C_2C_3B\to B_1C'B'}$.

After these steps, Alice holds C_1A and Bob has $B_1C'B'$. If $\Delta_1+\Delta_2$ is zero, the final state is exactly the pure state $\Phi_{C_1B_1}\otimes\varphi_{C'AB'R}$; the redistribution is perfectly accomplished since $\varphi_{C'AB'R}$ is the same pure state as $\varphi_{C''A''BR}$ but with A on Alice's side and $C'B'$ on Bob's side. In the process, $\log_2 d_{C_3}$ qubits are transmitted from Alice to Bob, $\log_2 d_{C_2}$ ebits are consumed and $\log_2 d_{C_1}$ ebits are distilled. Generally, the distance between the end state $\left(V\circ W^{-1}\right)\cdot\left(\Phi_{C_2A_2}\otimes\varphi_{C''A''BR}\right)$ and the pure state $\Phi_{C_1B_1}\otimes\varphi_{C'AB'R}$ in terms of the trace norm is not longer than $\Delta_1+\Delta_2$, which can be derived from (9) and (10).

QSR: i.i.d. case–At this stage, we apply our protocol to the i.i.d. case and show that $\Delta_1+\Delta_2$ approaches zero in the asymptotic region. Our technical result is summarized in a theorem:

Theorem 3 *(one-shot version of QSR) Suppose that quantum system A'' is a duplicate of A, B' is a duplicate of B, C' and C'' are duplicates of C, and $C=C_1C_2C_3$. Density operators $\Phi_{C_1B_1}$ and $\Phi_{C_2A_2}$ represent maximally entangled pure states. For any given pure states $\Upsilon_{initial}=\Phi_{C_2A_2}\otimes\varphi_{C''A''BR}$ and $\Upsilon_{final}=\Phi_{C_1B_1}\otimes\varphi_{C'AB'R}$, there are isometries $V_{C_2C_3B\to B_1C'B'}$ and $W_{C_1C_3A\to A_2C''A''}$ such that*

$$\left\|\left(V\circ W^{-1}\right)\cdot\Upsilon_{initial}-\Upsilon_{final}\right\|_1\le\Delta_1+\Delta_2, \qquad (11)$$
$$\left\|\left(W\circ V^{-1}\right)\cdot\Upsilon_{final}-\Upsilon_{initial}\right\|_1\le\Delta_1+\Delta_2, \qquad (12)$$

where $\varphi_{C''A''BR}$ and $\varphi_{C'AB'R}$ are the same pure state as φ_{CABR}, the upper bound $\Delta_1+\Delta_2$ equals $\gamma_1+\eta_1+\gamma_2+\eta_2$ that can be calculated from Eqs. (5) and (8).

Consider the i.i.d. case: Alice, Bob, and the reference system share n copies of the state $|\varphi_{CABR}\rangle$, i.e., $|\Psi_{C^nA^nB^nR^n}\rangle=|\varphi_{CABR}\rangle^{\otimes n}$ with C^nA^n on Alice's side and B^n on Bob's side, where we use the notion $A^n=A_1A_2\cdots A_n$. The task is to redistribute C^n from Alice to Bob. This is the redistribution of quantum states that have a tensor power structure.

We first recall the relevant knowledge about typical subspace [5, 6]. Based on quantum state $|\Psi_{C^nA^nB^nR^n}\rangle$ we can define δ-typical subspace of F^n, $F=A,B,C,AR,BR$, which is the subspace spanned by the eigenstates of $(\varphi_F)^{\otimes n}$ with the corresponding eigenvalues λ satisfying $2^{-n(S(F)+\delta)}\le\lambda\le2^{-n(S(F)-\delta)}$, where δ is a small positive number and $S(F)$ is von Neumann entropy of φ_F. The projector onto the δ-typical subspace of F^n is denoted by $\Pi^\delta_{F^n}$. We introduce the three new normalized states:

$$|\Omega_{C^{typ}A^nB^nR^n}\rangle\propto\Pi^\delta_{C^n}|\Psi_{C^nA^nB^nR^n}\rangle \qquad (13)$$
$$\left|\check{\Omega}_{C^{typ}B^n(AR)^n}\right\rangle\propto\Pi^\delta_{(AR)^n}\Pi^\delta_{B^n}\Pi^\delta_{C^n}|\Psi_{C^nA^nB^nR^n}\rangle \qquad (14)$$
$$\left|\hat{\Omega}_{C^{typ}A^n(BR)^n}\right\rangle\propto\Pi^\delta_{(BR)^n}\Pi^\delta_{A^n}\Pi^\delta_{C^n}|\Psi_{C^nA^nB^nR^n}\rangle \qquad (15)$$

where C^{typ} is used to denote the support of $\Pi^\delta_{C^n}$. For a sufficiently large n, we have

$$\left\|\Psi_{C^nA^nB^nR^n}-\Omega_{C^{typ}A^nB^nR^n}\right\|_1\le e^{-c\delta^2n} \qquad (16)$$

with a positive constant number c.

To redistribute C^n in $|\Psi_{C^nA^nB^nR^n}\rangle$ from Alice to Bob, Alice first makes a projective measurement on C^n with projectors $\Pi^\delta_{C^n}$ and $I_{C^n}-\Pi^\delta_{C^n}$. If the outcome of $I_{C^n}-\Pi^\delta_{C^n}$ is obtained, then the protocol fails. If the outcome of $\Pi^\delta_{C^n}$ is obtained, then the systems is in the state $|\Omega_{C^{typ}A^nB^nR^n}\rangle$. The inequality (16) indicates that the probability of failure vanishes when n approaches to infinite. When quantum state $\Omega_{C^{typ}A^nB^nR^n}$ is obtained, they then redistribute C^{typ} from Alice to Bob via our protocol for one-shot version of QSR with the reference states $\hat{\Omega}_{C^{typ}A^n(BR)^n}$ and $\check{\Omega}_{C^{typ}B^n(AR)^n}$.

We use $\Delta_1+\Delta_2$ to describe how well the redistribution is accomplished. A key point is that dimensions and

purities in the expression of $\Delta_1 + \Delta_2$ now have bounds that are related to entropy quantities [5, 6]. Assume that n is large enough and $C^{typ} = C_1^n C_2^n C_3^n$ with dimensions

$$\log_2 d_{C_1^n} = n \left[I\left(B;C\right) - 6t\delta \right]/2, \tag{17}$$

$$\log_2 d_{C_2^n} = n \left[I\left(A;C\right) - 6t\delta \right]/2, \tag{18}$$

$$\log_2 d_{C_3^n} = n \left[I\left(C;R\left|B\right.\right) + 12t\delta + 2\eta \right]/2, \tag{19}$$

for some $\eta \in [-t\delta, t\delta]$, where t is a constant bigger than one, $I\left(B;C\right)$ and $I\left(C;R\left|B\right.\right)$ are quantum mutual information and conditional mutual information defined on $\left|\varphi_{CABR}\right\rangle$ [19]. The exact value of η is determined by the dimension of C^{typ}. It can be shown for a large n

$$\Delta_1 + \Delta_2 \leq 4\left(e^{-c\delta^2 n} + \sqrt[4]{2 \times 2^{-n(2\eta+3t\delta)}} \right), \tag{20}$$

which goes to zero in the asymptotic region.

Note that $\Omega_{C^{typ}A^nB^nR^n}$ and $\Psi_{C^nA^nB^nR^n}$ have no difference when n goes to infinity [see (16)] and δ can be arbitrarily small, we can say that the redistribution of each copy of $\left|\varphi_{CABR}\right\rangle$ can be accomplished by sending $Q = I\left(C;R\left|B\right.\right)/2$ qubits from Alice to Bob, consuming $E_1 = I(A;C)/2$ ebits and distilling $E_2 = I(B;C)/2$ ebits in the asymptotic region. These are the optimal rates as shown in [15]. The reverse redistribution process can be accomplished by sending Q qubits from Bob to Alice, consuming E_2 ebits and distilling E_1 ebits per copy in the asymptotic region. The contents of these two processes can be expressed via a simple formula

$$\varphi_{AC|B} + E_1\left[qq\right] \overset{Q}{\Longleftrightarrow} \varphi_{A|CB} + E_2\left[qq\right], \tag{21}$$

where $\varphi_{AC|B}$ represents the quantum state φ_{CAB} with AC on Alice's side and B on Bob's side, $\varphi_{A|CB}$ represents the same quantum state φ_{CAB} but with A on Alice's side and BC on Bob's side, and $[qq]$ denotes an ebit between Alice and Bob [5]. We emphasize that the formula should be understood in the asymptotic region of the i.i.d. case. Also notably, E_1 and E_2 change into each other under the exchange of A and B, and Q is invariant under this exchange, namely, (21) is symmetric.

Summary–We have developed a protocol for one-shot version of QSR, which can lead to more natural resource rates in the i.i.d. case. The protocol is simplified from a combination of protocols for FQRS and FQSW problems. The simplification is based on a generalized decoupling theorem, which may have further applications in more complicated communication cases such as quantum state exchange [20] and multipartite communication. Since FQRS problem is the reverse of FQSW problem [6], the time-reversal symmetry is apparent in our protocol.

Note added–Recently Oppenheim studied the QSR from the viewpoint of coherent state-merging [21]. Our protocol presented here removes the decoupling operations, giving it the advantage of simplicity which makes it more promising to explore redistributing quantum states with structures other than the tensor power in the i.i.d. case.

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